Reciprocal transformations and flat metrics on Hurwitz spaces

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Abstract

We consider hydrodynamic systems which possess a local Hamiltonian structure. To such a system there are also associated an infinite number of nonlocal Hamiltonian structures. We give necessary and sufficient conditions so that, after a nonlinear transformation of the independent variables, the reciprocal system still possesses a local Hamiltonian structure. We show that, under our hypotheses, bi–hamiltonicity is preserved by the reciprocal transformation. Finally we apply such results to reciprocal systems of genus g Whitham-KdV modulation equations.

1 Introduction

Systems of hydrodynamic type that admit Riemann invariants are a class of quasilinear evolutionary PDEs of the form

$$u^i_t = v^i(\boldsymbol{u})u^i_x, \quad i = 1, \ldots, n, \tag{1}$$

where $\boldsymbol{u} = (u^1, \ldots, u^n)$ (see e.g. [5, 6, 7, 2, 25, 28, 29]). The systems (1) admit a local Hamiltonian structure if there exists a nondegenerate flat diagonal metric $g_{ii}(\boldsymbol{u})(du^i)^2$ solution to [29]

$$\partial_j \ln \sqrt{g_{ii}(\boldsymbol{u})} = \frac{\partial_j v^i(\boldsymbol{u})}{v^j(\boldsymbol{u}) - v^i(\boldsymbol{u})}. \tag{2}$$

The corresponding Hamiltonian operator

$$J^{ij}(\boldsymbol{u}) = g^{ii}(\ vu) \left(\delta^i_j \frac{d}{dx} - \Gamma^j_{ik}(\boldsymbol{u})u^k_x \right), \quad g^{ii} = 1/g_{ii}, \tag{3}$$

with $\Gamma^i_{jk}(\boldsymbol{u})$ are the Christoffel symbols of the metric $g_{ii}(\boldsymbol{u})$, defines a Poisson bracket on functionals

$$\{A, B\} = \int \frac{\delta A}{\delta u^i(x)} J^{ij} \frac{\delta A}{\delta u^k(x)} dx.$$

Such local Hamiltonian structures were introduced by Dubrovin-Novikov [5] and we refer to (3) as DN Hamiltonian structures. The Hamiltonian form of the equations (1) is

$$u^i_t = \{u^i, H\} = J^{ij}(\boldsymbol{u})\partial_j h(\boldsymbol{u}) = v^i(\boldsymbol{u})u^i_x, \quad i = 1, \ldots, n, \tag{4}$$

where $H = \int h(\boldsymbol{u})dx$ is the Hamiltonian. The system (4) possesses an infinite number of conservation laws and commuting flows and it is integrable through the generalized hodograph transform [29]. The formula (2) is crucial for the integrability property of diagonal Hamiltonian systems: if one interprets it as an overdetermined system on n unknown functions $v^i(\boldsymbol{u})$, ($g_{ii}(\boldsymbol{u})$ given), one can generate for any other solution $w^i(\boldsymbol{u})$, a symmetry $u^i_\tau = w^i(\boldsymbol{u})u^i_x$ of (4), namely $(u^i_t)_\tau = (u^i_\tau)_t$. One can prove the completeness property of this class of symmetries which implies integrability [29]. For any symmetry $w^i(\boldsymbol{u})$ of the Hamiltonian system (3), one can define the metric $\tilde{g}_{ii}(\boldsymbol{u}) = (w^i(\boldsymbol{u}))^2 g_{ii}(\boldsymbol{u})$, which is still flat and it is related to the metric $g_{ii}(\boldsymbol{u})$ by a Combescure transformation.

From a differential geometric point of view, a non-degenerate flat diagonal metric is equivalent to giving an orthogonal coordinate system on a flat space. Locally this coordinate system is parameterized by $n(n-1)/2$ functions of two variables. The problem of determining orthogonal coordinate systems dates back to the 19th century (see [31] and references therein). In the case $n = 2$ the problem has been solved classically, while it is still open for $n \geq 3$. Zakharov [31] showed that the problem can be solved by the dressing method.

All non–trivial examples of flat metrics have been obtained in the framework of the theory of Frobenius manifolds [2, 4], when the metric is of Egorov type. We recall that a metric $g_{ii}(\boldsymbol{u})$ is Egorov, if its rotation coefficients

$$\beta_{ij}(\boldsymbol{u}) \equiv \frac{\partial_i \sqrt{g_{jj}(\boldsymbol{u})}}{\sqrt{g_{ii}(\boldsymbol{u})}}, \quad i \neq j,$$

are symmetric, namely $\beta_{ij}(\boldsymbol{u}) = \beta_{ji}(\boldsymbol{u})$.

In this paper we address the problem of finding nontrivial examples of flat metric not of Egorov type, applying reciprocal transformations to the Hamiltonian structures of DN systems.

Indeed any DN system also possesses an infinite number of nonlocal Hamiltonian structures (see [12, 10, 21, 22]), since equation (2) defines $g_{ii}(\boldsymbol{u})$ up to a multiple $g_{ii}(\boldsymbol{u})/f^i(u^i)$, where $f^i(u^i)$ is an arbitrary function of u^i. Although the metric $g_{ii}(\boldsymbol{u})$ may happen to be flat for a particular choice of $f^i(u^i)$, it will not be flat in general. In particular, if the metric $g_{ii}(\boldsymbol{u})$ is of constant Riemannian curvature c or conformally flat with curvature tensor

$$R^{ij}_{ij}(\boldsymbol{u}) = w^i(\boldsymbol{u}) + w^j(\boldsymbol{u}), \quad i \neq j, \tag{5}$$

where the $w^i(\boldsymbol{u})$ satisfy (2), then the Hamiltonian operator associated to (1) is nonlocal and takes the special form

$$\begin{aligned} J^{ij}(\boldsymbol{u}) &= g^{ii}(\boldsymbol{u})\left(\delta^i_j \frac{d}{dx} - \Gamma^j_{ik}(\boldsymbol{u})u^k_x\right) + c u^i_x \left(\frac{d}{dx}\right)^{-1} u^j_x, \\ J^{ij}(\boldsymbol{u}) &= g^{ii}(\boldsymbol{u})\left(\delta^i_j \frac{d}{dx} - \Gamma^j_{ik}(\boldsymbol{u})u^k_x\right) + w^i(\boldsymbol{u})u^i_x \left(\frac{d}{dx}\right)^{-1} u^j_x + u^i_x \left(\frac{d}{dx}\right)^{-1} w^j(\boldsymbol{u})u^j_x, \end{aligned} \tag{6}$$

respectively. The first operator was introduced by Ferapontov and Mokhov [12], while the second one by Ferapontov[10].

Reciprocal transformations are a class of transformations of the independent variables and were introduced in gas dynamic [27]. Assuming that the DN hydrodynamic system (1) admits conservation laws

$$B(\boldsymbol{u})_t = A(\boldsymbol{u})_x, \quad N(\boldsymbol{u})_t = M(\boldsymbol{u})_x$$

with $B(\boldsymbol{u})M(\boldsymbol{u}) - A(\boldsymbol{u})N(\boldsymbol{u}) \neq 0$, then we can perform a change of the independent variables $(x,t) \to (\hat{x}(x,t,\boldsymbol{u}), \hat{t}(x,t,\boldsymbol{u})$ by the relations

$$d\hat{x} = B(\boldsymbol{u})dx + A(\boldsymbol{u})dt, \qquad d\hat{t} = N(\boldsymbol{u})dx + M(\boldsymbol{u})dt.$$

Then the reciprocal system

$$u^i_{\hat{t}} = \frac{B(\boldsymbol{u})v^i(\boldsymbol{u}) - A(\boldsymbol{u})}{M(\boldsymbol{u}) - N(\boldsymbol{u})v^i(\boldsymbol{u})} u^i_{\hat{x}} = \hat{v}^i(\boldsymbol{u}) u^i_{\hat{x}},$$

is clearly a system of hydrodynamic type.

Since reciprocal transformations send conservation laws to conservation laws, it is natural to investigate their effect on the corresponding Hamiltonian structure. The reciprocal metric is given by the formula

$$\hat{g}_{ii}(\boldsymbol{u}) = \left(\frac{M(\boldsymbol{u}) - N(\boldsymbol{u})v^i(\boldsymbol{u})}{B(\boldsymbol{u})M(\boldsymbol{u}) - A(\boldsymbol{u})N(\boldsymbol{u})} \right)^2 g_{ii}(\boldsymbol{u})$$

and clearly if $g_{ii}(\boldsymbol{u})$ is flat $\hat{g}_{ii}(\boldsymbol{u})$ is in general not flat. Linear reciprocal transformations, namely when $B(\boldsymbol{u}), A(\boldsymbol{u}), M(\boldsymbol{u})$ and $N(\boldsymbol{u})$ are constants, preserve flatness of the metric and locality of DN Hamiltonian operators (see Tsarev [29] and Pavlov [26]). In the case of nonlinear reciprocal transformations [13], Ferapontov and Pavlov have proven that the reciprocal to a flat metric is, in general, conformally flat. Moreover, Ferapontov [10] gave necessary and sufficient condition for the reciprocal to a flat metric to be a constant curvature in case the reciprocal transformation is a linear combination of Casimirs, momentum and the Hamiltonian density.

In a recent paper [1], we have proven that the Camassa–Holm (CH) modulation equations admit a local bi-hamiltonian structure of DN type and the corresponding flat metrics are reciprocal to the constant curvature and conformally flat metric of the Korteweg-de Vries (KdV) modulation equations. It is remarkable that none of the metrics of CH Hamiltonian structures are of Egorov type. For the above reasons, we are interested in a systematic investigation on the conditions under which the reciprocal to a (non)-flat metric is flat.

In this manuscript we work out necessary and sufficient conditions for the reciprocal metric to be flat, when the initial metric is either flat or constant curvature or conformally flat. The necessary and sufficient conditions for reciprocal flat metrics of sections 4-6 can be applied to search new examples of flat metrics on Hurwitz spaces. As a by-product we obtain non-trivial examples of flat metrics which are non-Egorov on the moduli space of hyperelliptic Riemann surfaces.

Finally, supposing that the initial system is bi-hamiltonian, namely it possesses two compatible Hamiltonian operators (see [20, 2, 4, 8, 11, 15, 23, 24]), we give sufficient conditions such that the reciprocal hydrodynamic system is bi-hamiltonian as well. We recall that bi–hamiltonicity is preserved by linear transformations [30].

The plan of the paper is as follows. In section 2 we set the notation and we compute the reciprocal Riemannian curvature tensor and the reciprocal Hamiltonian structure for any metric associated to the initial system. In section 3 we give sufficient conditions for the bi–hamiltonicity of the reciprocal to a bi–hamiltonian system when the transformation is nonlinear. In section 4 and 5 we consider the case of reciprocal transformations in x (respectively t) and we present the complete set of necessary and sufficient conditions for a reciprocal metric to be flat, when the initial metric is either flat or of constant curvature or conformally flat (respectively flat). In section 7 we consider reciprocal transformations of both variables x and t and we give sufficient conditions for a reciprocal metric to be flat, when the initial metric is either flat or of constant curvature or conformally flat. All of the necessary and sufficient conditions in sections 5–7 are expressed in Riemann invariants of the initial system and are compatible with the results in [10], where applicable.

Finally, in section 8, we give examples of flat reciprocal metrics on the moduli space of hyperelliptic Riemann surfaces. In particular, we relate by a reciprocal transformation the genus g Whitham–KdV hierarchy to the genus g Whitham–Camassa–Holm hierarchy.

2 The reciprocal Hamiltonian structure

In the following, we consider a DN Hamiltonian hydrodynamic system in Riemann invariants as in (3)

$$u^i_t = v^i(\boldsymbol{u})u^i_x. \tag{7}$$

Let $g^{ii}(\boldsymbol{u})$ be a non–degenerate metric such that for convenient $f^i(u^i)$, $i = 1, \dots, n$, $g^{ii}(\boldsymbol{u})f^i(u^i)$ is a flat metric associated to the local Hamiltonian operator of the system (7). Let $H_i(\boldsymbol{u})$, $\beta_{ij}(\boldsymbol{u})$ and $\Gamma^i_{jk}(\boldsymbol{u})$ be respectively the Lamé coefficients the rotation coefficients and the Christoffel symbol of a diagonal non-degenerate metric $g_{ii}(\boldsymbol{u})$ associated to (1),

$$H_i(\boldsymbol{u}) = \sqrt{g_{ii}(\boldsymbol{u})}, \qquad \beta_{ij}(\boldsymbol{u}) = \frac{\partial_i H_j(\boldsymbol{u})}{H_i(\boldsymbol{u})}, \quad i \neq j,$$

$$\Gamma^i_{jk}(\boldsymbol{u}) = \frac{1}{2} g^{im}(\boldsymbol{u}) \left(\frac{\partial g_{mk}(\boldsymbol{u})}{\partial u^j} + \frac{\partial g_{mj}(\boldsymbol{u})}{\partial u^k} - \frac{\partial g_{kj}(\boldsymbol{u})}{\partial u^m} \right),$$

then the nonzero elements of the Riemannian curvature tensor are

$$\begin{aligned} R^{ij}_{ik}(\boldsymbol{u}) &= -\frac{\partial_k \beta_{ij}(\boldsymbol{u}) - \beta_{ik}(\boldsymbol{u})\beta_{kj}(\boldsymbol{u})}{H_i(\boldsymbol{u})H_j(\boldsymbol{u})} \equiv 0, \qquad i \neq j \neq k \\ R^{ik}_{ik}(\boldsymbol{u}) &= -\frac{\Delta_{ik}(\boldsymbol{u})}{H_i(\boldsymbol{u})H_k(\boldsymbol{u})} \equiv \sum_{(l)} \epsilon^l w^i_{(l)}(\boldsymbol{u}) w^k_{(l)}(\boldsymbol{u}), \qquad i \neq k \end{aligned} \tag{8}$$

where $\epsilon^l = \pm 1$, $w^i_{(l)}(\boldsymbol{u})$ are affinors of the metric and

$$\Delta_{ik}(\boldsymbol{u}) = \partial_i \beta_{ik}(\boldsymbol{u}) + \partial_k \beta_{ki}(\boldsymbol{u}) + \sum_{m \neq i,k} \beta_{mi}(\boldsymbol{u})\beta_{mk}(\boldsymbol{u}),$$

and the Hamiltonian operator associated to $g^{ii}(\boldsymbol{u})$ is of nonlocal type [12, 10]

$$J^{ij}(\boldsymbol{u}) = g^{ii}(\boldsymbol{u})\left(\delta^i_j \frac{d}{dx} - \Gamma^j_{ik}(\boldsymbol{u})u^k_x\right) + \sum_l \epsilon^{(l)} w^i_{(l)}(\boldsymbol{u})u^i_x \left(\frac{d}{dx}\right)^{-1} w^j_{(l)}(\boldsymbol{u})u^j_x. \tag{9}$$

If $g^{ii}(\boldsymbol{u})$ is either flat or constant curvature or conformally flat, $R^{ij}_{ij}(\boldsymbol{u})$ is either zero or constant or as in (5) and $J^{ij}(\boldsymbol{u})$ takes the form (3) or (6), respectively.

Given conservation laws

$$B(\boldsymbol{u})_t = A(\boldsymbol{u})_x, \quad N(\boldsymbol{u})_t = M(\boldsymbol{u})_x$$

for the system (7), a reciprocal transformation of the independent variables x, t is defined by [27]

$$d\hat{x} = B(\boldsymbol{u})dx + A(\boldsymbol{u})dt, \qquad d\hat{t} = N(\boldsymbol{u})dx + M(\boldsymbol{u})dt. \tag{10}$$

Then the reciprocal system

$$u^i_{\hat{t}} = \hat{v}^i(\boldsymbol{u})u^i_{\hat{x}} = \frac{B(\boldsymbol{u})v^i(\boldsymbol{u}) - A(\boldsymbol{u})}{M(\boldsymbol{u}) - N(\boldsymbol{u})v^i(\boldsymbol{u})} u^i_{\hat{x}}, \tag{11}$$

is still Hamiltonian with $\hat{J}^{ij}(\boldsymbol{u})$ Hamiltonian operator associated to the reciprocal metric

$$\hat{g}_{ii}(\boldsymbol{u}) = \left(\frac{M(\boldsymbol{u}) - N(\boldsymbol{u})v^i(\boldsymbol{u})}{B(\boldsymbol{u})M(\boldsymbol{u}) - A(\boldsymbol{u})N(\boldsymbol{u})}\right)^2 g_{ii}(\boldsymbol{u}). \tag{12}$$

Let $\hat{H}_i(\boldsymbol{u})$, $\hat{\beta}_{ij}(\boldsymbol{u})$, $\hat{\Gamma}^i_{jk}(\boldsymbol{u})$ and $\hat{R}^{ij}_{km}(\boldsymbol{u})$, be respectively, the Lamé coefficients the rotation coefficients and the Christoffel symbol for the reciprocal metric $\hat{g}_{ii}(\boldsymbol{u})$. In the following we compute their expressions and that of the operator $\hat{J}^{ij}$. In [13], Ferapontov and Pavlov have characterized the tensor of the reciprocal Riemannian curvature and the reciprocal Hamiltonian structure when the initial metric $g_{ii}(\boldsymbol{u})$ is flat. To simplify notations, we drop the $\boldsymbol{u}$ dependence in the lengthy formulas.

Theorem 2.1 *Let $g^{ii}(\boldsymbol{u})$ be the contravariant diagonal metric as above for the Hamiltonian system (7) with Riemannian curvature tensor as in (8) or in (5). Then, for the contravariant reciprocal metric $\hat{g}^{ii}(\boldsymbol{u}) = 1/\hat{g}_{ii}(\boldsymbol{u})$ defined in (12), the only possible non-zero components of the reciprocal Riemannian curvature tensor are*

$$\begin{aligned} \hat{R}^{ik}_{ik}(\boldsymbol{u}) \quad &= \quad \frac{H_i H_k}{\hat{H}_i \hat{H}_k} R^{ik}_{ik} - (\nabla B)^2 + \frac{H_k}{\hat{H}_k}\nabla^k\nabla_k B + \frac{H_i}{\hat{H}_i}\nabla^i\nabla_i B - \hat{v}^k\hat{v}^i(\nabla N)^2 \\ &\quad + \hat{v}^k \frac{H_i}{\hat{H}_i}\nabla^i\nabla_i N + \hat{v}^i\frac{H_k}{\hat{H}_k}\nabla^k\nabla_k N - (\hat{v}^k + \hat{v}^i) < \nabla B, \nabla N >, \quad i \neq k \end{aligned} \tag{13}$$

where

$$< \nabla B(\boldsymbol{u}), \nabla N(\boldsymbol{u}) > = \sum_m g^{mm}(\boldsymbol{u})\partial_m B(\boldsymbol{u})\,\partial_m N(\boldsymbol{u}),$$

$$\nabla^i\nabla_i B(\boldsymbol{u}) = g^{ii}(\boldsymbol{u})\left(\partial^2_i B(\boldsymbol{u}) - \sum_m \Gamma^m_{ii}(\boldsymbol{u})\,\partial_m B(\boldsymbol{u})\right),$$

$$\nabla^i\nabla_j B(\boldsymbol{u}) = g^{ii}(\boldsymbol{u})\left(\partial_i\partial_j B(\boldsymbol{u}) - \Gamma^i_{ij}(\boldsymbol{u})\,\partial_i B(\boldsymbol{u}) - \Gamma^j_{ij}(\boldsymbol{u})\,\partial_j B(\boldsymbol{u})\right).$$

Proof. To compute the reciprocal Riemannian curvature tensor, we first compute the reciprocal rotation coefficients. Since the initial system is Hamiltonian, the rotation coefficients of the initial metric $g^{ii}(\boldsymbol{u})$ satisfy $\beta_{ik}(\boldsymbol{u}) = \dfrac{\partial_i H_k(\boldsymbol{u})}{H_i(\boldsymbol{u})} = \dfrac{\partial_i (H_k(\boldsymbol{u})v^k(\boldsymbol{u}))}{H_i(\boldsymbol{u})v^i(\boldsymbol{u})}$. Moreover $v_i(\boldsymbol{u}) = \dfrac{\partial_i M(\boldsymbol{u})}{\partial_i N(\boldsymbol{u})} = \dfrac{\partial_i A(\boldsymbol{u})}{\partial_i B(\boldsymbol{u})}$. Using the above expressions, it is straightforward to verify that the reciprocal rotation coefficients satisfy

$$
\begin{aligned}
\hat\beta_{ik}(\boldsymbol{u}) &\equiv \frac{\partial_i \hat H_k(\boldsymbol{u})}{\hat H_i(\boldsymbol{u})} \\
&= \beta_{ik} - \frac{(M-Nv_k)H_k\partial_i B}{H_i(BM-AN)} + \frac{(M-Nv_k)H_k\partial_i N(A-Bv_i)}{(BM-AN)(M-Nv_i)H_i} + \frac{H_k(v_i-v_k)\partial_i N}{(M-Nv_i)H_i} \\
&= \beta_{ik}(\boldsymbol{u}) - \hat H_k(\boldsymbol{u})\frac{\partial_i B(\boldsymbol{u})}{H_i(\boldsymbol{u})} - \hat H_k(\boldsymbol{u})\hat v_k(\boldsymbol{u})\frac{\partial_i N(\boldsymbol{u})}{H_i(\boldsymbol{u})}.
\end{aligned}
$$

For the diagonal metric $\hat g^{ii}(\boldsymbol{u})$ the only possibly non zero elements of the Riemannian curvature tensor are $\hat R^{ij}_{ik}(\boldsymbol{u})$, $(i\neq j\neq k\neq i)$ and $\hat R^{ij}_{ij}(\boldsymbol{u})$, $(i\neq j)$. To prove $\hat R^{ij}_{ik}(\boldsymbol{u})=0$, $(i\neq j\neq k\neq i)$, we use

$$
\begin{aligned}
&\partial_j\hat\beta_{ik}(\boldsymbol{u}) - \hat\beta_{ij}(\boldsymbol{u})\hat\beta_{jk}(\boldsymbol{u}) = \partial_j\left(\beta_{ik} - \hat H_k\frac{\partial_i B}{H_i} - \hat H_k\hat v_k\frac{\partial_i N}{H_i}\right) \\
&-\left(\beta_{ij} - \hat H_j\frac{\partial_i B}{H_i} - \hat H_j\hat v_j\frac{\partial_i N}{H_i}\right)\left(\beta_{jk} - \hat H_k\frac{\partial_j B}{H_j} - \hat H_k\hat v_k\frac{\partial_j N}{H_j}\right) \\
&= \partial_j\beta_{ik} - \beta_{ij}\beta_{jk} - \hat H_k\hat v_k H_i\nabla^i\nabla_j N(\boldsymbol{u}) - \hat H_k H_i\nabla^i\nabla_j B(\boldsymbol{u}),
\end{aligned}
$$

that is $\hat R^{ij}_{ik}(\boldsymbol{u})=0$ if and only if $\nabla^i\nabla_j B(\boldsymbol{u}) = 0 = \nabla^i\nabla_j N(\boldsymbol{u})$, $(i\neq j)$. Indeed, the Darboux equations

$$\partial_j\beta_{ik}(\boldsymbol{u}) - \beta_{ij}(\boldsymbol{u})\beta_{jk}(\boldsymbol{u}) = 0$$

are equivalent to the condition $R^{ij}_{ik}(\boldsymbol{u})=0$. By hypothesis, $\tilde g_{ii}(\boldsymbol{u}) = g_{ii}(\boldsymbol{u})/f^i(u^i)$ is a flat metric, then the Christoffel symbols of the two metrics satisfy $\tilde\Gamma^i_{ij}(\boldsymbol{u}) = \Gamma^i_{ij}(\boldsymbol{u})$, $(i\neq j)$, so that $\nabla^i\nabla_j B(\boldsymbol{u}) = 0 = \nabla^i\nabla_j N(\boldsymbol{u})$, $(i\neq j)$ and, finally, $\hat R^{ij}_{ik}(\boldsymbol{u})=0$.

To prove (13), we set

$$\hat R^{ik}_{ik}(\boldsymbol{u}) = -\frac{\hat\Delta_{ik}(\boldsymbol{u})}{\hat H_k(\boldsymbol{u})\hat H_i(\boldsymbol{u})}, \quad (i\neq k), \quad \text{where} \quad \hat\Delta_{ik}(\boldsymbol{u}) = \partial_i\hat\beta_{ik} + \partial_k\hat\beta_{ki} + \sum_{m\neq i,k}\hat\beta_{mi}\hat\beta_{mk}.$$

Then

$$
\begin{aligned}
\hat\Delta_{ik}(\boldsymbol{u}) = \Delta_{ik} &- \frac{\hat H_k}{H_i}\left(\partial_i^2 B - \sum_m \Gamma^m_{ii}\partial_m B\right) - \frac{\hat H_i}{H_k}\left(\partial_k^2 B - \sum_m \Gamma^m_{kk}\partial_m B\right) \\
&- \frac{\hat H_k\hat v_k}{H_i}\left(\partial_i^2 N - \sum_m \Gamma^m_{ii}\partial_m N\right) - \frac{\hat H_i\hat v_i}{H_k}\left(\partial_k^2 N - \sum_m \Gamma^m_{kk}\partial_m N\right) \\
&+ \sum_m \frac{\hat H_i\hat H_k}{H_m^2}\left((\partial_m B)^2 + \hat v_i\hat v_k(\partial_m N)^2 + (\hat v_i+\hat v_k)\partial_m B\,\partial_m N\right)
\end{aligned}
$$

$$= \Delta_{ik} - \hat{H}_k H_i \nabla^i \nabla_i B - \hat{H}_i H_k \nabla^k \nabla_k B - \hat{H}_k \hat{v}_k H_i \nabla^i \nabla_i N - \hat{H}_i \hat{v}_i H_k \nabla^k \nabla_k N$$

$$+ \hat{H}_i \hat{H}_k \left((\nabla B)^2 + \hat{v}_i \hat{v}_k (\nabla N)^2 + (\hat{v}_i + \hat{v}_k) < \nabla B \, , \, \nabla N > \right) .$$

from which (13) immediately follows. $\square$

We now compute the reciprocal affinors and the reciprocal Hamiltonian operator of a hydrodynamic system (7) with (nonlocal) Hamiltonian operator (6). To this aim, we introduce the auxiliary flows

$$u^i_\tau = n^i(\boldsymbol{u}) u^i_x = J^{ij}(\boldsymbol{u}) \partial_j N(\boldsymbol{u}), \qquad u^i_\zeta = b^i(\boldsymbol{u}) u^i_x = J^{ij}(\boldsymbol{u}) \partial_j B(\boldsymbol{u}), \tag{14}$$

$$u^i_{t_{(l)}} = w^i_{(l)}(\boldsymbol{u}) u^i_x = J^{ij}(\boldsymbol{u}) \partial_j H^{(l)}(\boldsymbol{u}),$$

respectively, generated by the densities of conservation laws associated to the reciprocal transformation (10), $B(\boldsymbol{u})$, $N(\boldsymbol{u})$, and by the densities of conservation laws $H^{(l)}(\boldsymbol{u})$ associated to the affinors $w^i_{(l)}$ of the Riemannian curvature tensor (5). By construction, all the auxiliary flows commute with (1). Introducing the following closed form

$$\begin{cases} d\hat{x} = B(\boldsymbol{u})dx + A(\boldsymbol{u})dt + P(\boldsymbol{u})d\tau + Q(\boldsymbol{u})d\zeta + \sum\limits_l T^{(l)}(\boldsymbol{u}) dt_{(l)}, \\ d\hat{t} = N(\boldsymbol{u})dx + M(\boldsymbol{u})dt + R(\boldsymbol{u})d\tau + S(\boldsymbol{u})d\zeta + \sum\limits_l Z^{(l)}(\boldsymbol{u}) dt_{(l)}, \\ d\hat{\tau} = d\tau, \quad d\hat{\zeta} = d\zeta, \quad d\hat{t}_{(l)} = dt_{(l)}, \end{cases} \tag{15}$$

where $P(\boldsymbol{u})$, $S(\boldsymbol{u})$, $Q(\boldsymbol{u})$, $R(\boldsymbol{u})$, $T^{(l)}(\boldsymbol{u})$, $Z^{(l)}(\boldsymbol{u})$ are defined up to additive constants, we have

$$\begin{aligned} v^i(\boldsymbol{u}) &= \frac{\partial_i A(\boldsymbol{u})}{\partial_i B(\boldsymbol{u})} = \frac{\partial_i M(\boldsymbol{u})}{\partial_i N(\boldsymbol{u})}, \qquad w^i_{(l)}(\boldsymbol{u}) = \frac{\partial_i T^{(l)}(\boldsymbol{u})}{\partial_i B(\boldsymbol{u})} = \frac{\partial_i Z^{(l)}(\boldsymbol{u})}{\partial_i N(\boldsymbol{u})}, \\ b^i(\boldsymbol{u}) &= \frac{\partial_i Q(\boldsymbol{u})}{\partial_i B(\boldsymbol{u})} = \frac{\partial_i S(\boldsymbol{u})}{\partial_i N(\boldsymbol{u})}, \qquad n^i(\boldsymbol{u}) = \frac{\partial_i P(\boldsymbol{u})}{\partial_i B(\boldsymbol{u})} = \frac{\partial_i R(\boldsymbol{u})}{\partial_i N(\boldsymbol{u})}. \end{aligned} \tag{16}$$

Inserting (16) into the right hand side of (14), we easily get

$$n^i(\boldsymbol{u}) = \nabla^i \nabla_i N + \sum_{(l)} \epsilon_{(l)} Z^{(l)} w^i_{(l)}, \quad b^i(\boldsymbol{u}) = \nabla^i \nabla_i B + \sum_{(l)} \epsilon_{(l)} T^{(l)} w^i_{(l)}, \tag{17}$$

Moreover, using (15), it is easy to verify that the reciprocal auxiliary flows

$$u^i_{\hat{\tau}} = \hat{n}^i(\boldsymbol{u}) u^i_{\hat{x}}, \qquad u^i_{\hat{\zeta}} = \hat{b}^i(\boldsymbol{u}) u^i_{\hat{x}}, \qquad u^i_{\hat{t}^{(l)}} = \hat{w}^i_{(l)}(\boldsymbol{u}) u^i_{\hat{x}},$$

satisfy

$$\begin{aligned} \hat{n}^i(\boldsymbol{u}) &= \left(n^i B - P + (N n^i - R) \hat{v}^i \right) = \left(\frac{H_i}{\hat{H}_i} n^i - P - \hat{v}^i R \right), \\ \hat{b}^i(\boldsymbol{u}) &= \left(b^i B - Q + (N b^i - S) \hat{v}^i \right) = \left(\frac{H_i}{\hat{H}_i} b^i - Q - \hat{v}^i S \right), \\ \hat{w}^i_{(l)}(\boldsymbol{u}) &= \left(w^i_{(l)} B - T^{(l)} + (N w^i_{(l)} - Z^{(l)}) \hat{v}^i \right) = \left(\frac{H_i}{\hat{H}_i} w^i_{(l)} - T^{(l)} - \hat{v}^i Z^{(l)} \right). \end{aligned} \tag{18}$$

Finally,

$$
\begin{gathered}
(\nabla B)^2 = 2Q - \sum_l \epsilon_{(l)} \left(T^{(l)}\right)^2, \qquad (\nabla N)^2 = 2R - \sum_l \epsilon_{(l)} \left(Z^{(l)}\right)^2, \\
< \nabla N, \nabla B >= \sum_l \epsilon_{(l)} T^{(l)} Z^{(l)} - P - S.
\end{gathered} \tag{19}
$$

Then inserting, (16-19) into the expression of $\hat{R}^{ij}_{ij}(\boldsymbol{u})$, we get the following

Theorem 2.2 *Let $g^{ii}(\boldsymbol{u})$ be the metric for the Hamiltonian hydrodynamic system (7), with Christoffel symbols $\Gamma^i_{jk}(\boldsymbol{u})$ and affinors $w^i_{(l)}$. Then, after the reciprocal transformation (10), the non zero components of the reciprocal Riemannian curvature tensor are*

$$
\hat{R}^{ij}_{ij}(\boldsymbol{u}) = \sum_l \epsilon^{(l)} \hat{w}^i_{(l)}(\boldsymbol{u}) \hat{w}^j_{(l)}(\boldsymbol{u}) + \hat{v}^i(\boldsymbol{u}) \hat{n}^j(\boldsymbol{u}) + \hat{v}^j(\boldsymbol{u}) \hat{n}^i(\boldsymbol{u}) + \hat{b}^i(\boldsymbol{u}) + \hat{b}^j(\boldsymbol{u}), \qquad i \neq j,
$$

and the reciprocal Hamiltonian operator takes the form

$$
\begin{aligned}
\hat{J}^{ij}(\boldsymbol{u}) \quad = \quad & \hat{g}^{ii}(\boldsymbol{u}) \left(\delta^i_j \frac{d}{d\hat{x}} - \hat{\Gamma}^j_{ik}(\boldsymbol{u}) u^k_{\hat{x}} \right) + \sum_l \epsilon^{(l)} \hat{w}^i_{(l)}(\boldsymbol{u}) u^i_{\hat{x}} \left(\frac{d}{d\hat{x}} \right)^{-1} \hat{w}^j_{(l)}(\boldsymbol{u}) u^j_{\hat{x}} \\
& + \hat{b}^i(\boldsymbol{u}) u^i_{\hat{x}} \left(\frac{d}{d\hat{x}} \right)^{-1} u^j_{\hat{x}} + u^i_{\hat{x}} \left(\frac{d}{d\hat{x}} \right)^{-1} \hat{b}^j(\boldsymbol{u}) u^j_{\hat{x}} \\
& + \hat{n}^i(\boldsymbol{u}) u^i_{\hat{x}} \left(\frac{d}{d\hat{x}} \right)^{-1} \hat{v}^j(\boldsymbol{u}) u^j_{\hat{x}} + \hat{v}^i(\boldsymbol{u}) u^i_{\hat{x}} \left(\frac{d}{d\hat{x}} \right)^{-1} \hat{n}^j(\boldsymbol{u}) u^j_{\hat{x}},
\end{aligned} \tag{20}
$$

where the reciprocal metric $\hat{g}^{ii}(\boldsymbol{u}) = 1/\hat{g}_{ii}(\boldsymbol{u})$ and the reciprocal affinors $\hat{n}^i(\boldsymbol{u})$, $\hat{b}^i(\boldsymbol{u})$ and $\hat{w}^i_{(l)}(\boldsymbol{u})$ have been defined in (12) and (18), respectively.

Corollary 2.3 *In the special case, when the reciprocal transformation changes only x ($N(\boldsymbol{u}) = 0$ and $M(\boldsymbol{u}) = 1$ in (10)), then the nonzero components of the transformed curvature tensor take the form*

$$
\begin{aligned}
\hat{R}^{ij}_{ij}(\boldsymbol{u}) \quad & = B^2(\boldsymbol{u}) R^{ij}_{ij}(\boldsymbol{u}) + B(\boldsymbol{u})(\nabla^i \nabla_i B(\boldsymbol{u}) + \nabla^j \nabla_j B(\boldsymbol{u})) - (\nabla B(\boldsymbol{u}))^2 \\
& = \sum_l \epsilon^{(l)} \hat{w}^i_{(l)}(\boldsymbol{u}) \hat{w}^j_{(l)}(\boldsymbol{u}) + \hat{b}^i(\boldsymbol{u}) + \hat{b}^j(\boldsymbol{u}).
\end{aligned} \tag{21}
$$

In the special case, when the reciprocal transformation changes only t ($B(\boldsymbol{u}) = 1$ and $A(\boldsymbol{u}) = 0$ in (10)), then the nonzero components of the transformed curvature tensor satisfy

$$
\begin{aligned}
\hat{R}^{ij}_{ij}(\boldsymbol{u}) \quad & = \quad \frac{M^2 R^{ij}_{ij} + M\,(v^j\, \nabla^i \nabla_i N + v^i\, \nabla^j \nabla_j N) - v^i\, v^j\, (\nabla N)^2}{(M - N v^i)(M - N v^j)} \\
& = \quad \sum_l \epsilon^{(l)} \hat{w}^i_{(l)}(\boldsymbol{u}) \hat{w}^j_{(l)}(\boldsymbol{u}) + \hat{v}^i(\boldsymbol{u}) \hat{n}^j(\boldsymbol{u}) + \hat{v}^j(\boldsymbol{u}) \hat{n}^i(\boldsymbol{u}).
\end{aligned} \tag{22}
$$

3 On bi–hamiltonicity of the reciprocal system

Bi–hamiltonicity [20] is a relevant property for a Hamiltonian system (see [2, 4, 8] and references therein). In this section, we suppose that the initial hydrodynamic system (7) $u^i_t = v^i(\boldsymbol{u})u^i_x$, $i = 1, \dots, n$, possesses a bi–hamiltonian structure, that is, it possesses two non–degenerate compatible Poisson structure $J^{ij}_\alpha(\boldsymbol{u})$, $\alpha = 1, 2$ and prove that the reciprocal system is still bi–hamiltonian.

We recall that the Poisson structures $J^{ij}_1(\boldsymbol{u})$ and $J^{ij}_2(\boldsymbol{u})$ are compatible if the linear combination

$$J^{ij}_1(\boldsymbol{u}) + \lambda J^{ij}_2(\boldsymbol{u})$$

is a non degenerate Poisson structure for arbitrary constant λ. Let us suppose that the diagonal metrics $g^{ii}_{(\alpha)}$ are associated to the Poisson structures $J^{ij}_\alpha(\boldsymbol{u})$, $\alpha = 1, 2$ of the form (3) or (6). If the second metric $g^{ii}_{(2)}$ is of the form $g^{ii}_{(2)} = g^{ii}_{(1)} f^i(u^i)$, where $f^i(u^i)$ is an arbitrary function of one variable, then $J^{ij}_1(\boldsymbol{u}) + \lambda J^{ij}_2(\boldsymbol{u})$ is a Poisson operator associated to the metric [24]

$$g^{ii}_{(1)}(\boldsymbol{u}) + \lambda g^{ii}_{(2)}(\boldsymbol{u}),$$

for arbitrary constants λ.

A natural question is whether, reciprocal transformations preserve the compatibility of Poisson brackets. Under the action of a linear reciprocal transformation, $\hat{t} = bx + at$, $\hat{x} = nx + mt$, with a, b, m, n constants such that $(b\,m - a\,n) \neq 0$, the reciprocal to a local Hamiltonian structure is local (see [29, 26]) and bi–hamiltonicity is preserved [24],[30].

Theorem 3.1 *Suppose that the hydrodynamic system (7),* $u^i_t = v^i(\boldsymbol{u})u^i_x$, $i = 1, \dots, n$, *possesses a bi–hamiltonian structure such that the associated metrics are non–singular, diagonal and of the form* $g^{ii}_{(1)}$ *and* $g^{ii}_{(2)} = g^{ii}_{(1)} f^i(u^i)$. *Then, after the transformation (12), the reciprocal system* $u^i_{\hat{t}} = \hat{v}^i(\boldsymbol{u})u^i_{\hat{x}}$, $i = 1, \dots, n$, *still possesses a (possibly non–local) bi–hamiltonian structure.*

To prove the theorem it is sufficient to show that the corresponding transformed Poisson operators $\hat{J}^{ij}_1(\boldsymbol{u})$ and $\hat{J}^{ij}_2(\boldsymbol{u})$ of the form (20) are compatible namely

$$\hat{J}^{ij}_\lambda = \hat{J}^{ij}_1(\boldsymbol{u}) + \lambda \hat{J}^{ij}_2(\boldsymbol{u}),$$

is an Hamiltonian operator associated to the metric

$$\hat{g}^{ii}_\lambda = \hat{g}^{ii}_{(1)} + \lambda \hat{g}^{ii}_{(1)} f^i(u^i). \tag{23}$$

It is straightforward to show that the local part of the operator $\hat{J}^{ij}_\lambda$ is linear in λ. In order to show that the nonlocal part is also linear is λ it is sufficient to use a result of [24] which says that the curvature tensor of a metric of the form $\hat{g}^{ii}_\lambda$ defined in (23) is linear in λ.

In the next sections we consider hydrodynamic type system (7) with a Hamiltonian structure associated to either a flat or a constant curvature or a conformally flat metric $g^{ii}_{(\alpha)}(\boldsymbol{u})$, $\alpha = 1, 2$, and we give conditions for the flatness of the reciprocal metric $\hat{g}^{ii}_{(\alpha)}(\boldsymbol{u})$. If $\hat{g}^{ii}_{(\alpha)}(\boldsymbol{u})$, $\alpha = 1, 2$ are both flat and the initial system is bi–hamiltonian, then, by the theorem above, the reciprocal system possesses a flat bi–hamiltonian structure.

4 Conditions for reciprocal flat metrics when only x changes

In this and the following sections, we suppose that the initial hydrodynamic system $u^i_t = v^i(\boldsymbol{u})u^i_x$, $i = 1,\dots,n$ is Hamiltonian as in (7) and the associated Hamiltonian operator $J^{ij}(\boldsymbol{u})$ is as in (3) or in (6), and we look for necessary and sufficient conditions such that, after a reciprocal transformation of type (10), one of the reciprocal metrics be flat. Since the reciprocal transformation $d\hat{x} = B(\boldsymbol{u})dx + A(\boldsymbol{u})dt$, $d\hat{t} = N(\boldsymbol{u})dx + M(\boldsymbol{u})dt$, is the composition of a transformation of the variable x and of a transformation of the variable t, we start our investigation with reciprocal transformations in which only the x variable changes.

In this section, using notations settled in sections 2 and 3, we give the complete set of necessary and sufficient conditions for the reciprocal metric to be flat, when $n \geq 3$ and the initial non–singular metric $g^{ii}(\boldsymbol{u})$ is either flat or of constant curvature c or conformally flat with affinors $w^i(\boldsymbol{u})$. Then the extended reciprocal transformation (15) is

$$d\hat{x} = B(\boldsymbol{u})dx + A(\boldsymbol{u})dt + P(\boldsymbol{u})d\tau + Q(\boldsymbol{u})d\zeta + T(\boldsymbol{u})dt_w,\ \hat{t} = t,\ \hat{\tau} = \tau;\ \hat{t}_w = t_w. \qquad (24)$$

Remark 4.1 *Let $b^i(\boldsymbol{u})$ and $w^i(\boldsymbol{u})$ be as in (14). The quantities $Q(\boldsymbol{u})$ and $T(\boldsymbol{u})$ in (24) satisfy the relations*

$$b^i(\boldsymbol{u}) = \frac{\partial_i Q(\boldsymbol{u})}{\partial_i B(\boldsymbol{u})}, \quad w^i(\boldsymbol{u}) = \frac{\partial_i T(\boldsymbol{u})}{\partial_i B(\boldsymbol{u})}.$$

Since (24) is a closed form, $Q(\boldsymbol{u})$ and $T(\boldsymbol{u})$ satisfy the relation

$$Q(\boldsymbol{u}) = \frac{1}{2}\Big(\nabla B(\boldsymbol{u})\Big)^2 + B(\boldsymbol{u})T(\boldsymbol{u}). \qquad (25)$$

Note that

- $T(\boldsymbol{u}) = 0$ *if $g^{ii}(\boldsymbol{u})$ is flat;*
- $T(\boldsymbol{u}) = \frac{c}{2}B(\boldsymbol{u})$ *if $g^{ii}(\boldsymbol{u})$ is of constant curvature c.*

After the reciprocal transformation (24), the metric $\hat{g}^{ii}(\boldsymbol{u})$ reciprocal to $g^{ii}(\boldsymbol{u})$, is flat if and only if the r.h.s. in (21) is zero, $\forall i,k = 1,\dots,n$, $i \neq k$, that is

$$B^2(\boldsymbol{u})(w^i(\boldsymbol{u}) + w^k(\boldsymbol{u})) + B(\boldsymbol{u})(\nabla^i\nabla_i B(\boldsymbol{u}) + \nabla^k\nabla_k B(\boldsymbol{u})) - (\nabla B(\boldsymbol{u}))^2 \equiv \hat{b}^i(\boldsymbol{u}) + \hat{b}^k(\boldsymbol{u}) = 0. \qquad (26)$$

(26) depends only on the initial Poisson structure and on the density of conservation law $B(\boldsymbol{u})$ in the reciprocal transformation. The above formula also shows that the class of metrics which are either flat or of constant curvature or conformally flat is left invariant by reciprocal transformations of the independent variable x and that $\hat{g}^{ii}(\boldsymbol{u})$ is flat if and only if the reciprocal affinor $\hat{b}^i(\boldsymbol{u}) \equiv 0$, $i = 1,\dots,n$. The next theorem gives the necessary and sufficient conditions for $\hat{b}^i(\boldsymbol{u}) \equiv 0$ in function of the initial system.

Theorem 4.2 *Let the contravariant non–singular diagonal metric $g^{ii}(\boldsymbol{u})$ associated to the initial system (7) be either flat or of constant curvature c or conformally flat with affinors*

$w^i(\boldsymbol{u})$. Let $d\hat{x} = B(\boldsymbol{u})dx + A(\boldsymbol{u})dt$, $d\hat{t} = dt$, with $B(\boldsymbol{u}) \not\equiv const.$. Then the reciprocal metric $\hat{g}^{ii}(\boldsymbol{u})$ is flat if and only if there exists a constant κ such that

$$\frac{Q(\boldsymbol{u})}{B(\boldsymbol{u})} \equiv \frac{(\nabla B(\boldsymbol{u}))^2}{2B(\boldsymbol{u})} + T(\boldsymbol{u}) = \kappa, \tag{27}$$

where $Q(\boldsymbol{u})$ and $T(\boldsymbol{u})$ are as in Remark 4.1. If in (27) $\kappa = 0$, then $B(\boldsymbol{u})$ is a Casimir associated to the metric $g^{ii}(\boldsymbol{u})$; if $\kappa \neq 0$ in (27) then $B(\boldsymbol{u})$ is proportional to a density of momentum associated to the metric $g^{ii}(\boldsymbol{u})$.

Proof. Since

$$\partial_i \left(\nabla B(\boldsymbol{u})\right)^2 = 2\partial_i B(\boldsymbol{u})\, \nabla^i \nabla_i B(\boldsymbol{u}), \tag{28}$$

(26) is equivalent to

$$\begin{aligned} 0 &= B^2(\boldsymbol{u})w^i(\boldsymbol{u}) + B(\boldsymbol{u})\nabla^i\nabla_i B(\boldsymbol{u}) - \frac{1}{2}\left(\nabla B(\boldsymbol{u})\right)^2 = B(\boldsymbol{u})b^i(\boldsymbol{u}) - Q(\boldsymbol{u}) \\ &= \frac{B^2(\boldsymbol{u})}{\partial_i B(\boldsymbol{u})}\partial_i \left(\frac{Q(\boldsymbol{u})}{B(\boldsymbol{u})}\right), \qquad i = 1, \dots, n, \end{aligned} \tag{29}$$

where $Q(\boldsymbol{u})$ is as in (25) and statement (27) immediately follows.

Finally, inserting (27) into the expression of the auxiliary flow $b^i(\boldsymbol{u})$, we get

$$b^i(\boldsymbol{u}) \equiv \frac{\partial_i Q(\boldsymbol{u})}{\partial_i B(\boldsymbol{u})} = \kappa, \qquad i = 1, \dots, n. \qquad \square$$

(27) settles quite restrictive conditions on the density of conservation law $B(\boldsymbol{u})$ in the reciprocal transformation for which we may hope that the reciprocal metric be flat. The following theorem shows that, conversely, if $B(\boldsymbol{u})$ is either a Casimir or a density of momentum associated to the metric $g^{ii}(\boldsymbol{u})$ then the reciprocal metric $\hat{g}^{ii}(\boldsymbol{u})$ is, at worse, of constant curvature.

Theorem 4.3 *Let the contravariant non–singular diagonal metric $g^{ii}(\boldsymbol{u})$ associated to the initial system (7) be either flat or of constant curvature c or conformally flat with affinors $w^i(\boldsymbol{u})$ and $T(\boldsymbol{u})$ as in Remark 4.1. Let $B(\boldsymbol{u})$ be either a Casimir ($b = 0$) or a density of momentum ($b = 1$) associated to the metric $g^{ii}(\boldsymbol{u})$. Then, under the reciprocal transformation $d\hat{x} = B(\boldsymbol{u})dx + A(\boldsymbol{u})dt$, $d\hat{t} = dt$, the reciprocal metric $\hat{g}^{ii}(\boldsymbol{u})$ is either flat or of constant curvature $\hat{c}$ where*

$$\hat{c} = 2bB(\boldsymbol{u}) - 2B(\boldsymbol{u})T(\boldsymbol{u}) - \left(\nabla B(\boldsymbol{u})\right)^2. \tag{30}$$

Proof. Let $B(\boldsymbol{u})$ be as in the hypothesis, then (21) becomes

$$\begin{aligned} \hat{R}^{ik}_{ik}(\boldsymbol{u}) &= B^2(\boldsymbol{u})\Big(w^i(\boldsymbol{u}) + w^k(\boldsymbol{u})\Big) + B(\boldsymbol{u})\Big(\nabla^i\nabla_i B(\boldsymbol{u}) + \nabla^k\nabla_k B(\boldsymbol{u})\Big) - \Big(\nabla B(\boldsymbol{u})\Big)^2 \\ &= 2bB(\boldsymbol{u}) - 2B(\boldsymbol{u})T(\boldsymbol{u}) - \Big(\nabla B(\boldsymbol{u})\Big)^2, \end{aligned}$$

and, for $l = 1, \dots, n$, we have

$$\partial_l \left(2bB(\boldsymbol{u}) - 2B(\boldsymbol{u})T(\boldsymbol{u}) - \Big(\nabla B(\boldsymbol{u})\Big)^2\right) = 2B(\boldsymbol{u})\Big(w^l(\boldsymbol{u})\partial_l B(\boldsymbol{u}) - \partial_l T(\boldsymbol{u})\Big) = 0,$$

from which we conclude that $\hat{R}^{ik}_{ik}(\boldsymbol{u})$ is a constant function. $\square$

The above necessary and sufficient conditions take a particular simple form in the case in which the initial metric is flat:

Corollary 4.4 *Let $g^{ii}(\boldsymbol{u})$ be the flat metric for DN system (3) and $d\hat{x} = B(\boldsymbol{u})dx + A(\boldsymbol{u})dt$ be a reciprocal transformation. Then the reciprocal metric $\hat{g}^{ii}(\boldsymbol{u}) = B^2(\boldsymbol{u})g^{ii}(\boldsymbol{u})$ is flat if and only if one of the following conditions hold true:*

1) B and A are constant functions;

2) $B(\boldsymbol{u})$ is a Casimir of the metric $g^{ii}(\boldsymbol{u})$ and $(\nabla B(\boldsymbol{u}))^2 = 0$;

3) $B(\boldsymbol{u})$ is a density of momentum for the metric $g^{ii}(\boldsymbol{u})$ and $(\nabla B(\boldsymbol{u}))^2 = 2B(\boldsymbol{u})$.

Remark 4.5 *If $B(\boldsymbol{u})$ and $N(\boldsymbol{u})$ are non trivial independent Casimirs of the flat metric $g^{ii}(\boldsymbol{u})$ and $(\nabla B(\boldsymbol{u}))^2 \neq 0$, then there exist a constant α and $A(\boldsymbol{u})$ such that, under the reciprocal transformation $d\hat{x} = (\alpha B(\boldsymbol{u}) + N(\boldsymbol{u}))dx + A(\boldsymbol{u})dt$, the reciprocal metric $\hat{g}^{ii}(\boldsymbol{u}) = (\alpha B(\boldsymbol{u}) + N(\boldsymbol{u}))^2 g^{ii}(\boldsymbol{u})$ is flat.*

If $B(\boldsymbol{u})$ is a density of momentum for the flat metric $g^{ii}(\boldsymbol{u})$ and $(\nabla B(\boldsymbol{u}))^2 - 2B(\boldsymbol{u}) = 2\alpha$, then under the reciprocal transformation $d\hat{x} = (B(\boldsymbol{u}) + \alpha)dx + A(\boldsymbol{u})dt$, the reciprocal metric $\hat{g}^{ii}(\boldsymbol{u}) = (B(\boldsymbol{u}) + \alpha)^2 g^{ii}(\boldsymbol{u})$ is flat.

5 Conditions for reciprocal flat metrics when only t changes

In this section, we give the complete set of necessary and sufficient conditions for a flat reciprocal metric, when $n \geq 3$, $g^{ii}(\boldsymbol{u})$ is the flat metric associated to the initial DN hydrodynamic type system

$$u^i_t = v^i(\boldsymbol{u})u^i_x = J^{ij}(\boldsymbol{u})\partial_i H(\boldsymbol{u}), \tag{31}$$

and the reciprocal transformation is $\hat{x} = x$, $d\hat{t} = N(\boldsymbol{u})dx + M(\boldsymbol{u})dt$. Under these hypotheses, the reciprocal metric $\hat{g}^{ii}(\boldsymbol{u})$ is flat if and only if the r.h.s. in (22) is identically zero $\forall i,k,\ i \neq k$, that is

$$M(\boldsymbol{u}) \Big(v^j(\boldsymbol{u})\, \nabla^i \nabla_i N(\boldsymbol{u}) + v^i(\boldsymbol{u})\, \nabla^j \nabla_j N(\boldsymbol{u}) \Big) - v^i(\boldsymbol{u})\, v^j(\boldsymbol{u})\, (\nabla N)^2(\boldsymbol{u}) \equiv 0. \tag{32}$$

(32) explicitly depends on the initial Poisson structure, on the density of conservation law $N(\boldsymbol{u})$ in the reciprocal transformation and on the density of Hamiltonian $H(\boldsymbol{u})$ associated to the metric $g^{ii}(\boldsymbol{u})$.

Theorem 5.1 *Let $g^{ii}(\boldsymbol{u})$ be the diagonal non–degenerate flat metric for (3) and let $v^i(\boldsymbol{u}) \neq$ const., $i = 1, \ldots, n$. Let $d\hat{t} = N(\boldsymbol{u})dx + M(\boldsymbol{u})dt$, $d\hat{x} = dx$, with $N(\boldsymbol{u}) \not\equiv$ const.. Then the reciprocal metric $\hat{g}^{ii}(\boldsymbol{u})$ is flat if and only if there exists a constant κ such that*

$$\frac{(\nabla N(\boldsymbol{u}))^2}{2M(\boldsymbol{u})} = \kappa. \tag{33}$$

If $\kappa = 0$, then $N(\boldsymbol{u})$ is a Casimir associated to the metric $g^{ii}(\boldsymbol{u})$; if $\kappa \neq 0$, then $N(\boldsymbol{u}) = \kappa H(\boldsymbol{u})$, where $H(\boldsymbol{u})$ is a density of Hamiltonian associated to the metric $g^{ii}(\boldsymbol{u})$.

Proof. (32) is equivalent to

$$\begin{aligned} 0 &= M(\boldsymbol{u})\Big(v^k(\boldsymbol{u})n^i(\boldsymbol{u})+v^i(\boldsymbol{u})n^k(\boldsymbol{u})\Big)-v^i(\boldsymbol{u})v^k(\boldsymbol{u})\left(\nabla N(\boldsymbol{u})\right)^2 \\ &= \frac{M^2(\boldsymbol{u})}{\partial_i N(\boldsymbol{u})\partial_k N(\boldsymbol{u})}\left(\partial_i M(\boldsymbol{u})\partial_k\left(\frac{\Big(\nabla N(\boldsymbol{u})\Big)^2}{2M(\boldsymbol{u})}\right)+\partial_k M(\boldsymbol{u})\partial_i\left(\frac{\Big(\nabla N(\boldsymbol{u})\Big)^2}{2M(\boldsymbol{u})}\right)\right), \end{aligned}$$

$\forall i,k=1,\dots,n,\ i\neq k$. If (33) holds, then $\hat R^{ik}_{ik}\equiv 0,\ i,k=1,\dots,n,\ i\neq k$, and $n^i(\boldsymbol{u})$ is either the null velocity flow ($N(\boldsymbol{u})$ Casimir of the initial metric) or

$$n^i(\boldsymbol{u})\equiv\frac{\partial_i\Big(\nabla N(\boldsymbol{u})\Big)^2}{2\partial_i N(\boldsymbol{u})}=\kappa\frac{\partial_i M(\boldsymbol{u})}{\partial_i N(\boldsymbol{u})}=\kappa v^i(\boldsymbol{u}),\qquad i=1,\dots,n.$$

Viceversa, suppose that the reciprocal metric is flat and $\Big(\nabla N(\boldsymbol{u})\Big)^2\neq 0$, then $\forall i,k,l=1,\dots,n,\ i\neq k$, it is straightforward to verify

$$\begin{aligned} 0 &\equiv \partial_l\left(M(\boldsymbol{u})\Big(v^i(\boldsymbol{u})n^k(\boldsymbol{u})+v^k(\boldsymbol{u})n^i(\boldsymbol{u})\Big)-v^i(\boldsymbol{u})v^k(\boldsymbol{u})\Big(\nabla N(\boldsymbol{u})\Big)^2\right) \\ &= \partial_l M(\boldsymbol{u})\Big(v^i(\boldsymbol{u})n^k(\boldsymbol{u})+v^k(\boldsymbol{u})n^i(\boldsymbol{u})\Big)-v^i(\boldsymbol{u})v^k(\boldsymbol{u})\Big(\nabla N(\boldsymbol{u})\Big)^2 \\ &= -v^i(\boldsymbol{u})v^k(\boldsymbol{u})\Big(\nabla N(\boldsymbol{u})\Big)^2\partial_l\log\left(\frac{\Big(\nabla N(\boldsymbol{u})\Big)^2}{M(\boldsymbol{u})}\right). \quad\Box \end{aligned}$$

Equation (33) settles quite restrictive conditions on the density of conservation law $N(\boldsymbol{u})$ in order to preserve flatness of the metric. The following theorem shows that, conversely, if $N(\boldsymbol{u})$ is either a Casimir or a density of Hamiltonian associated to the flat metric $g^{ii}(\boldsymbol{u})$ then the reciprocal metric $\hat g^{ii}(\boldsymbol{u})$ is either flat or associated to an hypersurface in the Euclidean space.

Theorem 5.2 *Let $g^{ii}(\boldsymbol{u})$ be a contravariant flat non–singular diagonal metric for (31) and let $N(\boldsymbol{u})$ be either a Casimir or a density of Hamiltonian associated to the metric $g^{ii}(\boldsymbol{u})$. Then, under the reciprocal transformation $d\hat t=N(\boldsymbol{u})dx+M(\boldsymbol{u})dt$, $d\hat x=dx$, the reciprocal metric $\hat g^{ii}(\boldsymbol{u})$ is either flat or the reciprocal Poisson operator takes the form*

$$\hat J^{ij}(\boldsymbol{u})=\hat g^{ii}(\boldsymbol{u})\delta^j_i\frac{d}{d\hat x}-\hat g^{ii}(\boldsymbol{u})\hat\Gamma^j_{ik}u^k_{\hat x}+\gamma\hat v^i u^i_{\hat x}\left(\frac{d}{d\hat x}\right)^{-1}\hat v^j_x u^j_x, \tag{34}$$

with γ constant.

Proof. If $N(\boldsymbol{u})$ is a Casimir and $\Big(\nabla N(\boldsymbol{u})\Big)^2=\gamma$, then $\hat R^{ik}_{ik}(\boldsymbol{u})=-\gamma\hat v^i(\boldsymbol{u})\hat v^k(\boldsymbol{u})$, $\forall i,k=1,\dots,n,\ i\neq k$ and the assertion easily follows.

If $N(\boldsymbol{u})$ is a density of Hamiltonian and $M(\boldsymbol{u})=\frac{1}{2}\Big(\nabla N(\boldsymbol{u})\Big)^2+\gamma$, then $\hat R^{ik}_{ik}(\boldsymbol{u})=\gamma\hat v^i(\boldsymbol{u})\hat v^k(\boldsymbol{u})$, $\forall i,k=1,\dots,n,\ i\neq k$, and the assertion easily follows.$\Box$

Remark 5.3 *If $B(\boldsymbol{u})$ and $N(\boldsymbol{u})$ are non trivial independent Casimirs of the flat metric $g^{ii}(\boldsymbol{u})$ and $(\nabla N(\boldsymbol{u}))^2 \neq 0$, then there exist a constant α and $M(\boldsymbol{u})$ such that, under the reciprocal transformation $d\hat{t} = (\alpha N(\boldsymbol{u}) + B(\boldsymbol{u}))dx + M(\boldsymbol{u})dt$, the reciprocal metric $\hat{g}^{ii}(\boldsymbol{u})$ is flat.*

If $N(\boldsymbol{u})$ is a density of Hamiltonian for the flat metric $g^{ii}(\boldsymbol{u})$ and $(\nabla N(\boldsymbol{u}))^2 - 2M(\boldsymbol{u}) = 2\alpha$, then under the reciprocal transformation $d\hat{t} = N(\boldsymbol{u})dx + \Big(M(\boldsymbol{u}) + \alpha\Big)dt$, the reciprocal metric $\hat{g}^{ii}(\boldsymbol{u})$ is flat.

6 Conditions for flat reciprocal metrics when the transformation changes both x and t

Let the initial hydrodynamic system be Hamiltonian as in (7)

$$u^i_t = v^i(\boldsymbol{u})u^i_x = J^{ij}(\boldsymbol{u})\partial_i H(\boldsymbol{u}), \tag{35}$$

where $J^{ij}(\boldsymbol{u})$ the Hamiltonian operator as in (3) or in (6) is associated to a initial metric $g^{ii}(\boldsymbol{u})$ either flat or of constant curvature c or conformally flat with affinors $w^i(\boldsymbol{u})$.

The reciprocal transformation

$$d\hat{x} = B(\boldsymbol{u})dx + A(\boldsymbol{u})dt, \qquad d\hat{t} = N(\boldsymbol{u})dx + M(\boldsymbol{u})dt,$$

is the composition of the following two reciprocal transformations of one variable

$$d\tilde{x} = B(\boldsymbol{u})dx + A(\boldsymbol{u})dt, \qquad d\tilde{t} = dt,$$

$$d\hat{x} = d\tilde{x}, \qquad d\hat{t} = \tilde{N}(\boldsymbol{u})d\tilde{x} + \tilde{M}(\boldsymbol{u})d\tilde{t},$$

where

$$\tilde{N}(\boldsymbol{u}) = \frac{N(\boldsymbol{u})}{B(\boldsymbol{u})}, \qquad \tilde{M}(\boldsymbol{u}) = \frac{M(\boldsymbol{u})B(\boldsymbol{u}) - N(\boldsymbol{u})A(\boldsymbol{u})}{B(\boldsymbol{u})}. \tag{36}$$

In view of the results of the previous sections, it is natural to restrict the attention to the case in which $B(\boldsymbol{u})$ is either a Casimir or a momentum density associated to the metric $g_{ii}(\boldsymbol{u})$. Then, after the first reciprocal transformation, the metric $\tilde{g}_{ii}(\boldsymbol{u})$ is either flat ($\hat{c} = 0$) or of constant curvature $\hat{c} \neq 0$, where $\hat{c}$ is the expression in the right hand side of (30). Let $\hat{c} = 0$, then after the second reciprocal transformation, by Theorem 5.1 the metric $\hat{g}_{ii}(\boldsymbol{u})$ is flat if and only if there exists a constant $\tilde{\kappa}$ such that

$$\frac{\Big(\tilde{\nabla}\tilde{N}(\boldsymbol{u})\Big)^2}{2\tilde{M}(\boldsymbol{u})} = \tilde{\kappa}. \tag{37}$$

We want to express (37) in an equivalent way as a function of the initial metric $g^{ii}(\boldsymbol{u})$ and of the density of conservation laws $N(\boldsymbol{u})$ and $B(\boldsymbol{u})$.

Remark 6.1 *Let $n^i(\boldsymbol{u})$ and $w^i(\boldsymbol{u})$ be as in (14). The quantities $R(\boldsymbol{u})$ and $Z(\boldsymbol{u})$ in (15) satisfy the relations*

$$n^i(\boldsymbol{u}) = \frac{\partial_i R(\boldsymbol{u})}{\partial_i N(\boldsymbol{u})}, \quad w^i(\boldsymbol{u}) = \frac{\partial_i Z(\boldsymbol{u})}{\partial_i N(\boldsymbol{u})}.$$

Since (15) is a closed form, $R(\boldsymbol{u})$ and $Z(\boldsymbol{u})$ satisfy the relation

$$R(\boldsymbol{u}) = \frac{1}{2}\Big(\nabla N(\boldsymbol{u})\Big)^2 + N(\boldsymbol{u})Z(\boldsymbol{u}). \tag{38}$$

Note that

- $Z(\boldsymbol{u}) = 0$ *if* $g^{ii}(\boldsymbol{u})$ *is flat;*
- $Z(\boldsymbol{u}) = \frac{c}{2}N(\boldsymbol{u})$ *if* $g^{ii}(\boldsymbol{u})$ *is of constant curvature* c*.*

Inserting (36) into (37), we get

$$\begin{aligned}\tilde{\kappa}\left(M(\boldsymbol{u}) - \frac{N(\boldsymbol{u})}{B(\boldsymbol{u})}A(\boldsymbol{u})\right) - \frac{1}{2}\Big(\nabla N(\boldsymbol{u})\Big)^2 - \frac{N^2(\boldsymbol{u})}{2B^2(\boldsymbol{u})}\Big(\nabla B(\boldsymbol{u})\Big)^2 \\ + \frac{N(\boldsymbol{u})}{B(\boldsymbol{u})} < \nabla B(\boldsymbol{u}), \nabla N(\boldsymbol{u}) >= 0.\end{aligned} \tag{39}$$

If $\tilde{\kappa} = 0$ in (39), then either $N(\boldsymbol{u}) = \nu_1 B(\boldsymbol{u})$, $M(\boldsymbol{u}) = \nu_1 A(\boldsymbol{u}) + \nu_2$, with ν_1, ν_2 non–zero constants, or there exists a constant ν_3 such that

$$\frac{(\nabla N(\boldsymbol{u}))^2}{2N(\boldsymbol{u})} + Z(\boldsymbol{u}) = \nu_3. \tag{40}$$

Comparing (40) with (33) and (38), we conclude that $N(\boldsymbol{u})$ is either a Casimir ($\nu_3 = 0$) or proportional to a momentum density ($\nu_3 \neq 0$) for the initial metric $g^{ii}(\boldsymbol{u})$.

If $\tilde{\kappa} \neq 0$ in (39), then there exists of a constant ν_4 such that

$$\frac{(\nabla N(\boldsymbol{u}))^2}{2} + Z(\boldsymbol{u})N(\boldsymbol{u}) = \tilde{\kappa}M(\boldsymbol{u}) + \nu_4 N(\boldsymbol{u}),$$

that is $N(\boldsymbol{u})$ is the linear combination with constant coefficients of a Hamiltonian density $H(\boldsymbol{u})$ and a momentum density associated to the initial metric $g^{ii}(\boldsymbol{u})$.

In the next theorem we summarize the above discussion. We use the notations settled in remarks 4.1 and 6.1.

Theorem 6.2 *Let the non–singular metric $g^{ii}(\boldsymbol{u})$ of system (35) be either flat or of constant curvature c or conformally flat with affinors $w^i(\boldsymbol{u})$. Let $B(\boldsymbol{u})$ be either a Casimir ($b = 0$) or a momentum density ($b = 1$) associated to the metric $g_{ii}(\boldsymbol{u})$ and such that*

$$\frac{(\nabla B(\boldsymbol{u}))^2}{2B(\boldsymbol{u})} + T(\boldsymbol{u}) = b, \tag{41}$$

where $T(\boldsymbol{u})$ has been defined in Remark 4.1. Then, after the reciprocal transformation $d\hat{x} = B(\boldsymbol{u})dx + A(\boldsymbol{u})dt$, $d\hat{t} = N(\boldsymbol{u})dx + M(\boldsymbol{u})dt$, the reciprocal metric $\hat{g}^{ii}(\boldsymbol{u})$ is flat if and only if either there exist non–zero constants ν_1 and ν_2 such that $N(\boldsymbol{u}) = \nu_1 B(\boldsymbol{u})$, $M(\boldsymbol{u}) = \nu_1 A(\boldsymbol{u}) + \nu_2$ or there exist (possibly zero) constants ν_3, ν_4 such that

$$\frac{(\nabla N(\boldsymbol{u}))^2}{2} + Z(\boldsymbol{u})N(\boldsymbol{u}) = \nu_3 M(\boldsymbol{u}) + \nu_4 N(\boldsymbol{u}), \tag{42}$$

where $Z(\boldsymbol{u})$ has been defined in Remark 6.1. If (42) holds true then $N(\boldsymbol{u})$ is either a Casimir ($\nu_3 = \nu_4 = 0$) or a momentum density ($\nu_3 = 0, \nu_4 = 1$) or a Hamiltonian density ($\nu_3 = 1, \nu_4 = 0$) or a linear combination with constants coefficients of the Casimirs, momentum and Hamiltonian density for the initial metric $g^{ii}(\boldsymbol{u})$.

The above theorem is far from setting the whole set of necessary and sufficient conditions for the reciprocal metric to be flat. In next theorem we give another set of sufficient conditions for a flat reciprocal metric when $B(\boldsymbol{u}) = H(\boldsymbol{u})$ is the Hamiltonian density in (35). Again we use the notations settled in Remarks 4.1 and 6.1.

Theorem 6.3 *Let the non–singular metric $g^{ii}(\boldsymbol{u})$ of system (35) be either flat or of constant curvature c or conformally flat with affinors $w^i(\boldsymbol{u})$. Let $J^{ij}(\boldsymbol{u})$ be the Hamiltonian operator associated to $g^{ii}(\boldsymbol{u})$ and let*

$$u^i_t = v^i(\boldsymbol{u})u^i_x = J^{ij}(\boldsymbol{u})\partial_j B(\boldsymbol{u}).$$

Under the reciprocal transformation $d\hat{x} = B(\boldsymbol{u})dx + A(\boldsymbol{u})dt$, $d\hat{t} = N(\boldsymbol{u})dx + M(\boldsymbol{u})dt$, the reciprocal metric $\hat{g}^{ii}(\boldsymbol{u})$ is flat if there exists a constant ν_5 such that

$$A(\boldsymbol{u}) = \frac{1}{2}(\nabla B(\boldsymbol{u}))^2 + T(\boldsymbol{u})B(\boldsymbol{u}), \quad \frac{(\nabla N(\boldsymbol{u}))^2}{2N(\boldsymbol{u})} + Z(\boldsymbol{u}) = \nu_5, \tag{43}$$

where $T(\boldsymbol{u})$ and $Z(\boldsymbol{u})$ have been defined in Remark 4.1 and Remark 6.1 respectively. If (43) holds true then $N(\boldsymbol{u})$ is either a Casimir ($\nu_5 = 0$) or a momentum density ($\nu_5 = 1$) for the initial metric $g_{ii}(\boldsymbol{u})$.

Proof. Under the hypotheses of the theorem, after the first reciprocal transformation

$$d\tilde{x} = B(\boldsymbol{u})dx + A(\boldsymbol{u})dt, \qquad d\tilde{t} = dt,$$

the transformed metric $\tilde{g}^{ii}(\boldsymbol{u}) = g^{ii}(\boldsymbol{u})B^2(\boldsymbol{u})$ is conformally flat with curvature tensor

$$\tilde{R}^{ik}_{ik}(\boldsymbol{u}) = \tilde{b}^i(\boldsymbol{u}) + \tilde{b}^k(\boldsymbol{u}).$$

After the second reciprocal transformation

$$d\hat{x} = d\tilde{x}, \qquad d\hat{t} = \tilde{N}(\boldsymbol{u})d\tilde{x} + \tilde{M}(\boldsymbol{u})d\tilde{t},$$

with $\tilde{N}(\boldsymbol{u})$ and $\tilde{M}(\boldsymbol{u})$ as in (36), the transformed metric $\hat{g}^{ii}(\boldsymbol{u})$ has Riemannian curvature tensor

$$\hat{R}^{ik}_{ik}(\boldsymbol{u}) = \frac{\tilde{M}^2(\tilde{b}^i + \tilde{b}^k) + \tilde{M}\big(\tilde{b}^i\tilde{\nabla}^k\tilde{\nabla}_k\tilde{N} + \tilde{b}^k\tilde{\nabla}^k\tilde{\nabla}_k\tilde{N}\big) - \tilde{b}^i\tilde{b}^k\big(\tilde{\nabla}\tilde{N}\big)^2}{\big(\tilde{M} - \tilde{b}^k\tilde{N}\big)\big(\tilde{M} - \tilde{b}^i\tilde{N}\big)},$$

where we have dropped the $\boldsymbol{u}$ in the r.h.s.. The condition $\hat{R}^{ik}_{ik}(\boldsymbol{u}) \equiv 0$ is satisfied in the above relation if

$$\tilde{M}(\boldsymbol{u})\partial_k\tilde{N}(\boldsymbol{u}) + \frac{1}{2}\partial_k\Big(\tilde{\nabla}\tilde{N}(\boldsymbol{u})\Big)^2 - \frac{\Big(\tilde{\nabla}\tilde{N}(\boldsymbol{u})\Big)^2}{2\tilde{M}(\boldsymbol{u})}\partial_k\tilde{M}(\boldsymbol{u}) = 0. \tag{44}$$

Inserting (36) and $\tilde{g}^{ii}(\boldsymbol{u}) = g^{ii}(\boldsymbol{u})B^2(\boldsymbol{u})$ into (44), we get

$$\begin{aligned} M - \frac{AN}{B} + B\nabla^k\nabla_k N - N\nabla^k\nabla_k B - <\nabla B, \nabla N> + \frac{N}{B}\Big(\nabla B\Big)^2 \\ -\frac{B}{2}\left(\Big(\nabla N\Big)^2 + \frac{N^2}{B^2}\Big(\nabla B\Big)^2 - 2\frac{N}{B} <\nabla B, \nabla N>\right)\frac{Bb^k - A}{MB - NA} \equiv 0. \end{aligned} \tag{45}$$

Let $A(\boldsymbol{u})$ and $N(\boldsymbol{u})$ be as in (43), then

$$M(\boldsymbol{u}) = <\nabla B(\boldsymbol{u}), \nabla N(\boldsymbol{u})> + N(\boldsymbol{u})T(\boldsymbol{u}) + B(\boldsymbol{u})Z(\boldsymbol{u}) - \nu_5 B(\boldsymbol{u}),$$

and (45) is identically satisfied. $\square$

Remark 6.4 *If the initial metric of system (35) is flat, (43) is equivalent to*

$$A(\boldsymbol{u}) = \frac{1}{2}\Big(\nabla B(\boldsymbol{u})\Big)^2,$$

and $N(\boldsymbol{u})$ is either a Casimir such that $\big(\nabla N(\boldsymbol{u})\big)^2 = 0$ or $N(\boldsymbol{u})$ is a momentum density such that $\big(\nabla N(\boldsymbol{u})\big)^2 - 2\nu_5 N(\boldsymbol{u}) = 0$.

7 Examples: flat metrics on moduli space of hyperelliptic curves

Flat metrics on Hurwitz spaces have been studied by Dubrovin in the framework of Frobenius manifolds [3]. The metrics considered in [3] are of Egorov type (see [2, 3, 19] and references therein for the role of the algebro–geometric approach in the theory of hydrodynamic systems). In this section we restrict ourselves to the moduli space of hyperelliptic Riemann surfaces and on this space we derive flat metrics which are not of Egorov type.

Let us consider the hyperelliptic curves of genus g

$$\mathcal{C} := \{(z,w) \in \mathbb{C}^2 \,|w^2 = \prod_{k=1}^{2g+1}(\lambda - u^k)\}, \quad u^k \neq u^j, k \neq j. \tag{46}$$

The distinct parameters $u^1, \ldots, u^{2g+1}$ are the local coordinates on the moduli space of hyperelliptic curves. On the Riemann surface $\mathcal{C}$ we define the meromorphic bidifferential $W(P,Q)$ as

$$W(P,Q) := d_P d_Q \log E(P,Q) \tag{47}$$

where $E(P,Q)$ is the prime form [9]. $W(P,Q)$ is a symmetric bi-differential on $\mathcal{C} \times \mathcal{C}$ with second order pole at the diagonal $P = Q$ with biresidue 1 and the properties:

$$\oint_{\alpha_k} W(P,Q) = 0\ ; \quad \oint_{\beta_k} W(P,Q) = 2\pi i\, \omega_k(P)\ ; \quad k = 1, \ldots, g\ . \tag{48}$$

Here $\{\alpha_k, \beta_k\}_{k=1}^g$ is the canonical basis of cycles on $\mathcal{C}$ and $\{\omega_k(P)\}_{k=1}^g$ is the corresponding set of holomorphic differentials normalized by $\oint_{\alpha_l} \omega_k = \delta_{kl}$, $k, l = 1, \ldots, g$. The dependence of the bidifferential W on branch points of the Riemann surface is given by the Rauch variational formulas [17]:

$$\frac{dW(P,Q)}{du^j} = \frac{1}{2} W(P,P_j) W(Q,P_j)\ , \tag{49}$$

where $W(P,P_j)$ denotes the evaluation of the bidifferential $W(P,Q)$ at $Q = P_j$ with respect to the standard local parameter $x_j(Q) = \sqrt{\lambda(Q) - u^j}$ near the ramification point P_j :

$$W(P,P_j) := \left.\frac{W(P,Q)}{dx_j(Q)}\right|_{Q=P_j}\ . \tag{50}$$

We consider the Abelian differentials

$$dp_s(Q) = -\frac{1}{2s-1} \operatorname*{Res}_{P=\infty} \lambda(P)^{\frac{2s-1}{2}} W(Q,P), \quad s = 1, 2, \ldots, \tag{51}$$

which are normalized differentials of the second kind with a pole at infinity of order $2s$ and behaviour

$$dp_s(Q) = -\frac{dz}{z^{2s}} + \text{regular terms}, \quad Q \to \infty$$

where $z = 1/\sqrt{\lambda}$ is the local coordinate in the neighbourhood of infinity.

Theorem 7.1 *[3] The diagonal metrics*

$$g^0_{ii} = \operatorname*{Res}_{Q=P_i}\left[\frac{dp_1^2(P)}{d\lambda}\right](du^i)^2 = \frac{1}{2}(dp_1(u^i))^2(du^i)^2, \qquad g^1_{ii} = \frac{g^1_{ii}}{u^i}, \tag{52}$$

where dp_1 is the differential (51), are compatible flat metrics on the moduli space of hyperelliptic Riemann surfaces.

The correspondent flat coordinates of the metric g^0_{ii} are the following [3]

$$h_0 = -\operatorname*{Res}_{\infty}\lambda^{\frac{1}{2}}dp_1, \;\; r^a = \frac{1}{2\pi i}\oint_{\beta_a}dp_1, \;\; s^a_0 = -\frac{1}{2\pi i}\oint_{\alpha_a}\lambda dp_1, \;\; a = 1,\dots,g. \tag{53}$$

The flat coordinates of the metric g^1_{kk} are obtained by the relations

$$p_1(0), \;\; r^a = \frac{1}{2\pi i}\oint_{\beta_a}dp_1, \;\; s^a_1 = -\frac{1}{2\pi i}\oint_{\alpha_a}\log\lambda dp_1, \;\; a = 1,\dots,g. \tag{54}$$

We observe that the coordinates r^a, $a = 1,\dots,g$, are the common Casimirs of the metrics g^0_{ii} and g^1_{ii}.

Let J^{ij}_0 and J^{ij}_1 be the Hamiltonian operators associated to the metrics $(g^0)^{ii}$ and $(g^1)^{ii}$ respectively and let us consider the Hamiltonian densities

$$h_s = -\operatorname*{Res}_{\infty}\lambda^{\frac{2s+1}{2}}dp_1 \;\; s = 0,1,\dots,. \tag{55}$$

Then, the equations

$$u^i_{t_s} = J^{ij}_0\frac{\delta h_{s+1}}{\delta u^j} = J^{ij}_1\frac{\delta h_s}{\delta u^j} = v^i_{(s)}u^i_x, \;\; s = 0,1,\dots, \tag{56}$$

corresponds to the KdV-Whitham hierarchy with $t_0 = x$ and $t_1 = t$ [14], [18].

The metrics $\frac{g^0_{ii}}{(u^i)^s}$, $s = 2,3,\dots$, are not flat and the non-zero elements of the associated curvature tensor $R^{ij}_{ij}(\boldsymbol{u},s)$ are

$$R^{ij}_{ij}(\boldsymbol{u},s) = -\frac{1}{2\sqrt{g^0_{ii}g^0_{jj}}}\left(\sum_{k=1}^{2g+1}(u^k)^s\partial_{u^k}W(P_i,P_j) + \frac{s}{2}((u^i)^{s-1} + (u_j)^{s-1}))W(P_i,P_j)\right). \tag{57}$$

Formulas (57) hold true also for $s = 0,1$, where the r.h.s. identically vanishes (a proof may be found in [3],[16]).

In the next lemma we evaluate the curvature tensor (57) for $s = 2,3$.

Lemma 7.2 *The metrics*

$$g^2_{ii} = \frac{g^0_{ii}}{(u^i)^2}, \quad g^3_{ii} = \frac{g^0_{ii}}{(u^i)^3} \tag{58}$$

are constant curvature and conformally flat respectively. The nonzero elements of the curvature tensor (57) take the form

$$R^{ij}_{ij}(\boldsymbol{u}, s=2) = -\frac{1}{2}, \qquad R^{ij}_{ij}(\boldsymbol{u}, s=3) = -\frac{3}{2}\left(\frac{dp_2(P_i)}{dp_1(P_i)} + \frac{dp_2(P_j)}{dp_1(P_j)}\right), \tag{59}$$

with $dp_{1,2}$ as in (51).

Proof. Using the fact that (57) vanishes for $s = 0, 1$ we obtain

$$\begin{aligned}\sum_{k=1}^{2g+1}(u^k)^s\partial_{u^k}W(P_i,P_j) = \sum_{k\neq i,j}(u^k)^s\partial_k W(P_i,P_j) + \frac{(u^i)^s}{u^i-u^j}\sum_{k\neq i,j}(u^j-u^k)\partial_k W(P_i,P_j)\\ -\frac{(u^j)^s}{u^i-u^j}\sum_{k\neq i,j}(u^i-u^k)\partial_k W(P_i,P_j) + \frac{(u^i)^s-(u^j)^s}{u^i-u^j}W(P_i,P_j)\end{aligned}$$

Using (49) and the residue theorem we re-write the above in the form

$$\begin{aligned}\sum_{k=1}^{2g+1}(u^k)^s\partial_{u^k}W(P_i,P_j) = -\operatorname*{Res}_{P=P_i,P_j,\infty}\lambda(P)^s\frac{W(P_i,P)W(P_j,P)}{d\lambda(P)}\\ -\frac{(u^i)^s}{u^i-u^j}\operatorname*{Res}_{P=P_i,\infty}(\lambda(P_j)-\lambda(P))\frac{W(P_i,P)W(P_j,P)}{d\lambda(P)}\\ +\frac{(u^j)^s}{u^i-u^j}\operatorname*{Res}_{P=P_j,\infty}(\lambda(P_i)-\lambda(P))\frac{W(P_i,P)W(P_j,P)}{d\lambda(P)}\end{aligned}$$

The last two terms in the r.h.s. of the above expression are holomorphic at infinity so that

$$\begin{aligned}&\sum_{k=1}^{2g+1}(u^k)^s\partial_{u^k}W(P_i,P_j)) + \frac{s}{2}((u^i)^{s-1}+(u_j)^{s-1}))W(P_i,P_j)\\ &= -\operatorname*{Res}_{P=\infty}\lambda(P)^s\frac{W(P_i,P)W(P_j,P)}{d\lambda(P)}\\ &-(u^i)^s\operatorname*{Res}_{P=P_i}\left[\left((1+\frac{\lambda(P_j)-\lambda(P)}{u^i-u^j}\right)\frac{W(P_i,P)W(P_j,P)}{d\lambda(P)}\right]\\ &-(u^j)^s\operatorname*{Res}_{P=P_j}\left[\left((1+\frac{\lambda(P_i)-\lambda(P)}{u^j-u^i}\right)\frac{W(P_i,P)W(P_j,P)}{d\lambda(P)}\right]\end{aligned}$$

For $s = 0, 1$ the first term in the r.h.s of the above expression vanishes because it is holomorphic at infinity. Since for $s = 0, 1$ the curvature tensor $R^{ij}_{ij}(\boldsymbol{u}, s = 0, 1)$ is equal to zero, it follows that the last two terms of the above expression are identically zero. So we

conclude that the curvature tensor takes the form

$$
R^{ij}_{ij}(\boldsymbol{u}, s) = -\frac{\mathrm{Res}_{P=\infty}\left[\lambda(P)^s \dfrac{W(P_i,P)W(P_j,P)}{d\lambda(P)}\right]}{dp_i(u^i)dp_1(u^j)} \tag{60}
$$

$$
= \begin{cases} -\dfrac{1}{2} & \text{for } s=2 \\ -\dfrac{3}{2}\left(\dfrac{dp_2(P_i)}{dp_1(P_i)} + \dfrac{dp_2(P_j)}{dp_1(P_j)}\right) & \text{for } s=3. \end{cases} \tag{61}
$$

The lemma is proved.

As a first application of the theorems in Section 4 on sufficient conditions for a reciprocal metric to be flat, we consider the reciprocal transformation of x, $d\hat{x} = r^a dx + A^{(a,j)}dt_j$, where r^a, $a = 1, \ldots, g$, are Casimirs common to all the metrics g^s_{ii}, $u^i_{t_j} = v^i_{(j)}u^i_x$ is the j–th modulated flow of the KdV hierarchy (56) and $A^{(a,j)}$ makes the transformation closed. Then the following results can be obtained in a straightforward way applying theorem 4.2.

Theorem 7.3 *Let g^s_{ii}, $s = 0, 1, 2, 3$ be the metrics defined in (52) and (58). Then the reciprocal metrics*

$$
\frac{g^s_{ii}}{(r^a)^2}, \quad a = 1, \ldots, g, \; s = 0, 1, 2, 3, \tag{62}
$$

where r^a, $a = 1, \ldots, g$, are the Casimirs defined in (53), are flat compatible diagonal metrics.

Proof. In order to prove that the metrics (62) are flat, it is sufficient to verify the condition (27). For $s = 0$ and $s = 1$, the quantity T defined in remark (4.1) is equal to zero and the condition (27) takes the form

$$
\sum_{i=1}^{2g+1} \frac{(\partial_i r^a)^2 (u^i)^s}{g^0_{ii}} = 0, \quad s = 0, 1, \quad , a = 1, \ldots, g.
$$

In the following we prove the above relation. Using the variational formula (49) we obtain

$$
\partial_i r^a = \frac{1}{2} dp(P_i)\omega_a(P_i)
$$

so that

$$
\begin{aligned}
\sum_{i=1}^{2g+1} \frac{(\partial_i r^a)^2 (u^i)^s}{g^0_{ii}} &= \frac{1}{2}\sum_{i=1}^{2g+1} (u^i)^s \omega_a(P_i)^2 \\
&= \sum_{i=1}^{2g+1} \operatorname*{Res}_{P=P_i} \frac{\lambda^s(\omega_a(\lambda))^2}{d\lambda} = 0, \quad s = 0, 1, \quad a = 1, \ldots, g,
\end{aligned} \tag{63}
$$

because $\dfrac{\lambda^s(\omega(\lambda))^2}{d\lambda}$, $s = 0, 1$, is a differential with simple poles at the branch points P_i and regular at infinity and therefore, the sum of all its residue is equal to zero. For the metric

g^2_{ii} the condition (27) takes the form

$$\sum_{i=1}^{2g+1} \frac{(\partial_i r^a)^2 (u^i)^2}{g^0_{ii}} - \frac{1}{2}(r^a)^2 = 0, \quad s = 0, 1, \;\; a = 1, \dots, g. \tag{64}$$

To prove the above relation we use (63) and then evaluate the residue at infinity obtaining

$$\sum_{i=1}^{2g+1} \frac{(\partial_i r^a)^2 (u^i)^2}{g^0_{ii}} = \sum_{i=1}^{2g+1} \operatorname*{Res}_{P=P_i} \frac{\lambda^2 (\omega_a(\lambda))^2}{d\lambda} = -\operatorname*{Res}_{\lambda=\infty} \frac{\lambda^2 (\omega_a(\lambda))^2}{d\lambda} = \frac{1}{2}(r^a)^2$$

because of the Riemann bilinear relations

$$r^a = \operatorname*{Res}_{\lambda=\infty} p_1(\lambda)\omega_a(\lambda) = \operatorname*{Res}_{\lambda=\infty} \sqrt{\lambda}\omega_a(\lambda). \tag{65}$$

For the metric g^3_{ii} the condition (27) takes the form

$$\sum_{i=1}^{2g+1} \frac{(\partial_i r^a)^2 (u^i)^3}{g^0_{ii}} - \frac{3}{4\pi i} r^a \oint_{b_a} dp_2 = 0, \quad a = 1, \dots, g,$$

since $T = -\frac{3}{4\pi i} \oint_{b_a} dp_2$. To prove the above relation we use (63) and then evaluate the residue at infinity obtaining

$$\sum_{i=1}^{2g+1} \frac{(\partial_i r^a)^2 (u^i)^3}{g^0_{ii}} = -\operatorname*{Res}_{\lambda=\infty} \frac{\lambda^3 (\omega_a(\lambda))^2}{d\lambda} = \frac{3}{4\pi i} r^a \oint_{b_a} dp_2,$$

because of the Riemann bilinear relations (65) and

$$\frac{1}{2\pi i} \oint_{b_a} dp_2 = \operatorname*{Res}_{\lambda=\infty} p_2(\lambda)\omega_a(\lambda) = \frac{1}{3} \operatorname*{Res}_{\lambda=\infty} \lambda^{\frac{3}{2}} \omega_a(\lambda).$$

The theorem is proved. $\square$

As a second application of the theorems on sufficient conditions for the reciprocal metric to be flat, we consider the reciprocal transformation of x, $d\hat{x} = p_1(0)dx + A^{(p,l)}dt_l$, where $p_1(0)$ is the Casimir associated to $g^1_{ii}(\boldsymbol{u})$ which generates the modulated first negative KdV flow (index $l = -1$ in the transformation). In the case of genus $g = 1$, we showed in [1] that this reciprocal transformation relates the modulated first negative KdV flow and the modulated Camassa–Holm equations. Next theorem generalizes such relation to any genus and can be obtained in a straightforward way applying theorem 4.2.

Theorem 7.4 *Let g^s_{ii}, $s = 2, 3$ be the metrics defined in (58). Then the reciprocal metrics*

$$\frac{g^s_{ii}}{p_1(0)^2}, \; s = 2, 3, \tag{66}$$

where $p_1(0)$ is the Casimir for g^1_{ii} defined in (54) are flat compatible diagonal metrics.

Proof. In order to prove that the metrics (66) are flat, it is sufficient to verify the condition (27). For the metric g^2_{ii} the condition (27) takes the form

$$\sum_{i=1}^{2g+1} \frac{(\partial_i p_1(0))^2 (u^i)^2}{g^0_{ii}} - \frac{1}{2(p_1(0))^2} = k p_1(0), \tag{67}$$

where k is a constant. To prove the above relation we first observe that

$$\operatorname*{Res}_{\lambda=\infty} \lambda^{\frac{1}{2}} \Lambda_0(\lambda) = -2p_1(0),$$

where $\Lambda_0(\lambda)$ is a normalized third kind differential with simple pole in $(0, \pm\sqrt{\prod_{k=1}^{2g+1}(-u^k)})$ with residues ± 1 respectively. Applying (49) we deduce

$$\partial_i p_1(0) = \frac{1}{4} dp_1(u^i) \Lambda_0(u^i).$$

Then we reduce the sum in (67) to the evaluation of a residue

$$\sum_{i=1}^{2g+1} \frac{(\partial_i p_1(0))^2 (u^i)^2}{g^0_{ii}} = \sum_{i=1}^{2g+1} \operatorname*{Res}_{P=P_i} \frac{\lambda^2 (\Lambda_0(\lambda))^2}{4d\lambda} = -\operatorname*{Res}_{P=\infty} \frac{\lambda^2 (\Lambda_0(\lambda))^2}{4d\lambda} = \frac{1}{2}(p_1(0))^2. \tag{68}$$

From the above relation we conclude that (67) is satisfied with $k = 0$.

For the metric g^3_{ii} the condition (27) takes the form

$$\sum_{i=1}^{2g+1} \frac{(\partial_i p_1(0))^2 (u^i)^3}{g^0_{ii}} - 3p_1(0)p_2(0) = 0, \tag{69}$$

because $T = -\frac{3}{2} p_2(0)$. To prove the above relation we use (68) obtaining

$$\sum_{i=1}^{2g+1} \frac{(\partial_i p_1(0))^2 (u^i)^3}{g^0_{ii}} = -\operatorname*{Res}_{P=\infty} \frac{\lambda^3 (\Lambda_0(\lambda))^2}{4d\lambda} = 3p_1(0)p_2(0).$$

because $\operatorname{Res}_{\lambda=\infty} \lambda^{\frac{3}{2}} \Lambda_0(\lambda) = -6p_2(0)$. The above relation shows the validity of (69). $\square$

As a third example we consider the Casimir h_0 that generates the positive KdV modulated flows ($s = 1$ in (56)).

Lemma 7.5 *The metric*

$$\frac{g^0_{ii}}{h_0^2 u^i} \tag{70}$$

is flat with h_0 the Casimir for g^0_{ii} defined in (53). The metrics

$$\frac{g^0_{ii}}{h_0^2}, \quad \frac{g^0_{ii}}{(h_0 u^i)^2}, \tag{71}$$

are respectively constant curvature and conformally flat.

Proof. To prove the lemma we need the relation

$$\sum_{i=1}^{2g+1} \frac{(\partial_i h_0)^2 (u^i)^s}{g^0_{ii}} = \sum_{i=1}^{2g+1} (dp_1(u^i))^2 (u^i)^s = \sum_{i=1}^{2g+1} \operatorname*{Res}_{P=P_i} \frac{\lambda^s dp_i(\lambda)^2}{d\lambda}$$
$$= -\operatorname*{Res}_{\infty} \frac{\lambda^s dp_i(\lambda)^2}{d\lambda} = \begin{cases} -2, & \text{for } s=0 \\ 2h_0, & \text{for } s=1 \\ \frac{1}{2}h_0^2 + 2h_1, & \text{for } s=2 \end{cases} \tag{72}$$

where h_1 has been defined in (55).

In order to prove that the metric (70) is flat, it is sufficient to verify the condition (27), where h_0 is a density of momentum for the metric g^0_{ii}/u^i, that is there exists a constant k such that

$$\sum_{i=1}^{2g+1} \frac{(\partial_i h_0)^2 u^i}{g^0_{ii}} = k h_0$$

and comparing with (72) we find $k = 2$.

The relation (72) immediately implies that the metrics in (71) are respectively with constant curvature -2 and conformally flat with affinors $\tilde{v}^i$

$$\tilde{v}^i = h_0 \frac{\partial_i h_1}{\partial_i h_0} - h_1.$$

□

As a final example we consider a reciprocal transformation of x and t of the form

$$\begin{cases} d\hat{x} &= r^a dx + A^a dt \\ d\hat{t} &= h_0 dx + M dt \end{cases} \tag{73}$$

where h_0 and r^a are the Casimirs defined in (53) for the metric g^0_{ii} and A^a and M are the terms which make the above two 1-forms closed with respect to the first Whitham-KdV flow, that is

$$u^i_t = v^i_{(1)} u^i_x = (J^0)^{ij} \frac{\delta h_2}{\delta u^j}, i = 1, \ldots, 2g+1.$$

where the hamiltonian density h_2 is defined in (55).

Let $\hat{g}^0_{ii}$ be the transformed metric of g^0_{ii} given by the relation (12)

$$\hat{g}^0_{ii} = \left(\frac{M - h_0 v^i}{r^a M - A^a h_0} \right)^2 g^0_{ii}. \tag{74}$$

Theorem 7.6 *The reciprocal metrics*

$$\frac{\hat{g}^0_{ii}}{u^i}, \; \frac{\hat{g}^0_{ii}}{(u^i)^2}$$

with $\hat{g}^0_{ii}$ defined in (74) form a flat pencil of metrics.

To prove the statement we apply theorem 6.2 with $B = r^a$ and $N = h_0$.

(63) with $s = 1$ gives $b = 0$ in theorem 6.2 and (72) for $s = 1$ is equivalent to the flatness condition (53) of theorem 6.2, with $\nu_3 = 0, \nu_4 = 1$ and we conclude that the metric $\frac{\hat{g}^0_{ii}(\boldsymbol{u})}{u^i}$ is flat.

For the second metric, similarly, (64) gives $b = 0$ in theorem 6.2 and (72) for $s = 2$ is equivalent to the flatness condition (53) of theorem 6.2, with $\nu_3 = 1, \nu_4 = 0$ and we conclude that the metric $\frac{\hat{g}^0_{ii}(\boldsymbol{u})}{(u^i)^2}$ is flat.

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